COMBINATORICA 6 (4) (1986) 287—298

ON WELL-QUASI-ORDERING-FINITE GRAPHS BY IMMERSION

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Received 10 September 1984

It has been conjectured by Nash-Williams that the class of all graphs is well-quasi-ordered under the quasi-order $\preceqq$ defined by immersion. Two partial results are proved which support this conjecture. (i) The class of finite simple graphs G with $G \not\preceqq K_{2,3}$ is well-quasi-ordered by $\preceqq$,(ii) it is shown that a class of finite graphs is well-quasi-ordered by $\preceqq$ provided that the blocks of its members satisfy certain restrictive conditions. (In particular, this second result implies that $\preceqq$ is a well-quasi-order on the class of graphs for which each block is either complete or a cycle.)

1. Introduction

In the present paper only finite (undirected) graphs without loops and multiple edges are considered. All graph theoretical terms which are not defined here are used in the sense of Bondy—Murty [1]. For graphs G and H, we write $G_{is} \leqq H$ if there is a subgraph of H that is isomorphic to a subdivision of G. A result of J. B. Kruskal [6] which is known as the *tree theorem* states the following. If $T_1, T_2, \dots$ is an infinite sequence of (finite) trees, then there exist positive integers i, j such that $i<j$ and $T_{is} \leqq T_j$. A short proof of this theorem was obtained by Nash-Williams [10]. The tree theorem was first conjectured by A. Vázsonyi, together with another conjecture which is still open. This second conjecture was that, for every infinite sequence $G_1, G_2, \dots$ of graphs in which every vertex has degree $\leqq 3$, there exist positive integers i, j such that $i<j$ and $G_{is} \leqq G_j$. In [11], Nash-Williams suggested a more general conjecture concerning a relation $\preceqq$ which is defined as follows. For a graph G, $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively; let further, $P(G)$ denote the set of paths which are contained in G as subgraphs) and which have length at least one. For graphs G and H, an *immersion* $\varphi\colon G \to H$ is a mapping $\varphi\colon V(G) \cup E(G) \to V(H) \cup P(H)$ such that the following conditions hold.

(i) $\varphi(v) \in V(H)$, $\varphi(e) \in P(H)$ for all $v \in V(G)$ and $e \in E(G)$,

(ii) the restriction of φ to $V(G)$ is a one-to-one mapping,

(iii) for each $e=(v, w) \in E(G)$, $\varphi(e)$ is a path from $\varphi(v)$ to $\varphi(w)$ which contains no vertex $\varphi(u)$ other than $\varphi(v)$ and $\varphi(w)$ $(u \in V(G))$,

(iv) the paths $\varphi(e)$ $(e \in E(G))$ are pairwise edge-disjoint.

AMS subject classification (1980): 05 C 10

Let $G \preceq H$ if there exists an immersion $\varphi: G \to H$. In [11], Nash-Williams conjectured that, for every infinite sequence $G_1, G_2, \dots$ of graphs there exist positive integers i, j such that $i<j$ and $G_i \preceq G_j$. (See also Nash-Williams [12].) Note that $G_s\preceq H$ always implies $G \preceq H$; further, if G and H are either both trees or both graphs in which every vertex has degree $\leqq 3$, then $G \preceq H$ and $G_s\preceq H$ are equivalent statements. Thus the above conjecture of Nash-Williams, if true, implies both the tree theorem and Vázsonyi's (second) conjecture.

It is convenient (and common practice) to discuss the above conjectures in terms of well-quasi-ordered classes. A binary relation $\leqq$ on a class Q is called a *quasi-order* if $\leqq$ is reflexive and transitive. For example, the relations ${}_s\preceq$ and $\preceq$ are quasi-orders on the class of all graphs. A class Q is *well-quasi-ordered (wqo) by* $\leqq$ if $\leqq$ is a quasi-order and if, for every infinite sequence $q_1, q_2, \dots$ of elements of Q, there are $i, j \in \mathrm{N}$ such that $i<j$ and $q_i \leqq q_j$. (We shall also refer to the pair $(Q, \leqq)$ as a quasi-order or a well-quasi-order (wqo). N denotes the set of positive integers.)

One possible way to obtain (at least) partial results on the conjecture of Nash-Williams is that of studying classes of graphs G for which $G \not\succeq H$ holds for some fixed graph H. A similar approach was made by Mader [9] for a related quasi-order ${}_m\preceq$ which is also known as the *minor inclusion relation* (terminology of [15]). *H is a minor of G* ($H_m \preceq G$) if a copy of H can be obtained from a subgraph of G by contracting some of its edges. Mader [9] proved that the class of all graphs G with $G \not\succeq_m K_5^-$ is wqo by ${}_m\preceq$. (K_5^- is the graph that results from the complete graph K_5 by deletion of an edge.) Hünseler [4] extended Mader's result by showing for several other (but finitely many) planar graphs H that the class of graphs G with $G \not\succeq_m H$ is wqo by ${}_m\preceq$. Recently, Robertson and Seymour made great progress on the problem of well-quasi-ordering by ${}_m\preceq$ (see e.g. [13, 14, 15]). One of the main results of these authors is that, for each planar graph H, the class of graphs G with $G \not\succeq_m H$ is wqo by ${}_m\preceq$. In the present paper the following is proved.

Theorem 1. *The class of graphs G with $G \not\succeq K_{2,3}$ is wqo by $\preceq$.*

Some of the methods used for the proof of this theorem are similar to those of Mader [9]. In addition, a second result is proved which states that a class of graphs is wqo by $\preceq$ provided that the blocks of its members have certain properties as specified in Section 4. Further, we remark that it is easy to construct an infinite sequence of graphs $G_1, G_2, \dots$ such that (i) $G_{i\,s} \not\preceq G_j$ for each $i, j \in \mathrm{N}$ ($i<j$) and (ii) each block of G_i is a triangle ($i \in \mathrm{N}$). (See e.g. [5, 15] for similar constructions.) In particular, this shows that the class of graphs considered in Theorem 1 is not wqo by ${}_s\preceq$.

2. Notations and preliminary results

Our notations are essentially the same as those of Bondy—Murty [1]. Thus there are just a few conventions that need some explanation. The letter G always denotes a graph. The vertex set of a graph is assumed to be non-empty. $\gamma(v, G)$ denotes the degree of v in G $(v \in V(G))$. A path P with $V(P)=\{v_i : i=1, \dots, n\}$, $E(P)=\{(v_i, v_{i+1}) : i=1, \dots, n-1\}$ is sometimes written as $P=(v_1, v_2, \dots, v_n)$. Analogously, we write $(v_1, v_2, \ldots, v_n, v_{n+1}=v_1)$ for cycles. We shall consider graphs rather than isomorphism classes of graphs; in this sense, K_n $(K_{m,n})$ stands for a fixed complete (complete bipartite) graph $(n, m\in\mathbf{N})$. For a subgraph $H\subseteq G$, we will also write $G-H$ instead of $G-V(H)$. For $H\subseteq G$, let C be a component of $G-H$; let further $N:=\{v\in V(H): (v, x)\in E(G)$ for some $x\in V(C)\}$ and let B be the graph that is induced in G by the vertex set $V(C)\cup N$. Then B is called a *branch of H in G*. A *rooted graph* is a pair (A, a) where A is a graph and $a\in V(A)$; a is called the *root* of (A, a). If $\varphi: A\to B$ is an immersion, then $\varphi(A)$ denotes the graph that is the union of the vertices $\varphi(v)$ and the paths $\varphi(e)$ $(v\in V(A), e\in E(A))$. As in [1], isolated vertices of G are accepted as blocks of G.

Let $(Q, \leqq)$ be a quasi-order. Let FQ be the class of finite sequences of elements of Q, and let Q^* be the class of finite subsets of Q. For members $a_1, \ldots, a_k$ and $b_1, \ldots, b_n$ of FQ, let $a_1, \ldots, a_k\leqq^F b_1, \ldots, b_n$ if there is a subsequence $b_{i(1)}, \ldots, b_{i(k)}$ of $b_1, \ldots, b_n$ such that $i(1)<\ldots<i(k)$ and $a_m\leqq b_{i(m)}$ for $m=1, \ldots, k$. Similarly, for $A, B\in Q^*$ let $A\leqq^* B$ if there exists a one-to-one mapping φ of A into B such that $a\leqq\varphi(a)$ for each $a\in A$. Further, let $(Q_i, \leqq_i)$ $(i=1, 2)$ be quasi-orders, and let $Q_1\times Q_2$ be the Cartesian product. For $(a_1, a_2), (b_1, b_2)\in Q_1\times Q_2$, let $(a_1, a_2)\leqq_1\wedge\leqq_2 (b_1, b_2)$ if $a_i\leqq_i b_i$ $(i=1, 2)$. The relations $\leqq^F$, $\leqq^*$, and $\leqq_1\wedge\leqq_2$ are quasi-orders on FQ, Q^*, and $Q_1\times Q_2$, respectively. A sequence q_i $(i\in\mathbf{N})$ of elements of Q is called a *good sequence of* $(Q, \leqq)$ if there are $j, m\in\mathbf{N}$ such that $j<m$ and $q_j\leqq q_m$; otherwise, it is called a *bad sequence of* $(Q, \leqq)$. Let $\pi: Q\to\mathbf{N}\cup\{0\}$ be a mapping. q_i $(i\in\mathbf{N})$ is called a *minimal bad sequence of* $(Q, \leqq)$ *(with respect to* π*)* if the following conditions hold. (i) q_i $(i\in\mathbf{N})$ is a bad sequence of $(Q, \leqq)$, (ii) if p_i $(i\in\mathbf{N})$ is a sequence of elements of Q such that, for some $n\in\mathbf{N}$, $\pi(p_n)<\pi(q_n)$ and $p_i=q_i$ for each $i<n$, then p_i $(i\in\mathbf{N})$ is a good sequence of $(Q, \leqq)$. Let S be a finite set. For $a, b\in S$, define $a\leqq_{tr} b$ iff $a=b$. Then S is wqo by $\leqq_{tr}$. The relation $\leqq_{tr}$ is called the *trivial well-quasi-order on* S.

In Lemmas 1 and 2, we collect some well-known results on well-quasi-ordering. For the proof of Lemma 1, see e.g. Higman [3, Theorem 4.3] and Nash-Williams [10]. Lemma 2 can be proved by standard arguments. (See e.g. Nash-Williams [10] or Mader [9].) For the reader's convenience, we give a proof of Lemma 2 (b).

Lemma 1. Let $(Q, \leqq)$, $(Q_i, \leqq_i)$ $(i=1, 2)$ *be well-quasi-orders. Then*

(a) $Q_1\times Q_2$ *is wqo by* $\leqq_1\wedge\leqq_2$,

(b) Q^* *is wqo by* $\leqq^*$,

(c) FQ *is wqo by* $\leqq^F$. ∎

Lemma 2. *Let* $(Q, \leqq)$ *be a quasi-order which is not a well-quasi-order and let* $\pi: Q\to\mathbf{N}\cup\{0\}$ *be a function.*

(a) *Then there exists a minimal bad sequence of* $(Q, \leqq)$ *with respect to* π.

(b) *Let* $q_i\in Q$ $(i=1, 2, \ldots)$ *be a minimal bad sequence of* $(Q, \leqq)$ *with respect to* π *and suppose that* $p_i\in Q$ $(i=1, 2, \ldots)$ *is a sequence for which there exists a function* $f: \mathbf{N}\to\mathbf{N}$ *such that* $p_i\leqq q_{f(i)}$ *and* $\pi(p_i)<\pi(q_{f(i)})$, $i=1, 2, \ldots$. *Then the set* $P:=\{p_i: i\in\mathbf{N}\}$ *is wqo by the restriction of* $\leqq$ *to* P.

Proof of Lemma 2 (b). Let $p_{i_k}\in P$, $k=1, 2, \ldots$. Pick $m\in\mathbf{N}$ such that $f(i_m)\leqq f(i_k)$ for each $k\geqq m$ and consider the sequence $q_1, q_2, \ldots, q_{f(i_m)-1}, p_{i_m}, p_{i_{m+1}}, \ldots$. This sequence is good since q_i $(i\in\mathbf{N})$ is a minimal bad sequence and $\pi(p_{i_m})<\pi(q_{f(i_m)})$. Consequently (since $q_i\leqq q_j$ $(i<j)$ is impossible) $q_j\leqq p_{i_k}$ for some j, k $(1\leqq j<f(i_m)$, $k\geqq m)$ or $p_{i_n}\leqq p_{i_l}$ for some n, l $(m\leqq n<l)$. But the former is also impossible since it would imply $q_j\leqq p_{i_k}\leqq q_{f(i_k)}$, $j<f(i_m)\leqq f(i_k)$. Hence we have proved that the sequence p_{i_k} $(k=1, 2, \ldots)$ is good. ∎

In the following we shall only consider minimal bad sequences of (possibly rooted or labelled) graphs where π is the function which assigns to a given graph its number of edges. Thus we shall always drop the suffix "with respect to π" thereby meaning that the sequence under consideration is minimal with respect to the edge number.

For a class $\mathscr{A}$ of graphs and a quasi-order $(Q, \leqq)$, let $\mathscr{A}(Q)$ be the class of pairs (A, λ) where $A\in\mathscr{A}$ and $\lambda\colon V(A)\to Q$ is a function. The members of $\mathscr{A}(Q)$ are called *labelled graphs*. For $(A, \lambda), (B, \mu)\in\mathscr{A}(Q)$, let $(A, \lambda)\trianglelefteq_{(Q,\leqq)}(B, \mu)$ if there exists an immersion $\varphi\colon A\to B$ such that $\lambda(v)\leqq\mu(\varphi(v))$ for each $v\in V(A)$. This defines a quasi-order on $\mathscr{A}(Q)$. Adopting the terminology of [15], we say that $\mathscr{A}$ is *well-behaved*, if, for each well-quasi-order $(Q, \leqq)$, the class $\mathscr{A}(Q)$ is wqo by $\trianglelefteq_{(Q,\leqq)}$.

3. The Proof of Theorem 1

In the following Lemmas 3, 4 and 5, the structure of graphs G with $G\not\trianglerighteq K_{2,3}$ is investigated. Lemmas 6 and 7 contain well-behavedness results on 2-connected graphs G with $G\not\trianglerighteq K_{2,3}$.

Lemma 3. *Let c be a cut vertex of a graph G and let B_1, B_2 be distinct blocks of G such that $c\in V(B_k)$ $(k=1, 2)$. If $\gamma(c, B_k)\geqq 3$ $(k=1, 2)$, then $G\trianglerighteq K_{2,3}$.*

Proof. Let $b_{k,j}\in V(B_k)$ $(j=1, 2, 3)$ be distinct neighbours of c $(k=1, 2)$. Since B_k-c is connected, we can find a path $P_k\subseteqq B_k-c$ that joins two of the vertices $b_{k,1}, b_{k,2}, b_{k,3}$ and avoids the third; w.l.o.g. let $b_{k,1}\notin P_k$ $(k=1, 2)$. Moreover, we can find a path $P_k'\subseteqq B_k-c$ that starts in $b_{k,1}$ and has just one vertex in common with P_k $(k=1, 2)$. Let $v_k=P_k\cap P_k'$ $(k=1, 2)$. Then $v_1\neq b_{1,m}$ for a certain $m\in\{2, 3\}$. One easily verifies that there exists an immersion $\varphi\colon K_{2,3}\to G$ which maps the vertices of $K_{2,3}$ onto $\{v_1, v_2, b_{1,1}, b_{1,m}, b_{2,1}\}$. ∎

A graph is *outerplanar* if it can be drawn in the (Euclidean) plane such that all vertices are on the boundary of the same face. Outerplanar graphs can be characterized as the graphs G for which $G\not\trianglerighteq_s K_4$ and $G\not\trianglerighteq_s K_{2,3}$. (See e.g. [2].)

Lemma 4. *Let G be a 2-connected graph with $K_4\not\cong G\not\trianglerighteq K_{2,3}$. Then*

(a) *$G\not\trianglerighteq_s K_4$ (and, consequently, G is outerplanar),*
(b) *G has just one Hamilton cycle C,*
(c) *if $(v, v_1), (v, v_2)\in E(G)\setminus E(C)$ $(v_1\neq v_2)$, then $(v_1, v_2)\in E(C)$,*
(d) *$\gamma(v, G)\leqq 4$ for each $v\in V(G)$.*

Proof. (a) and (b) are easy (and well-known) consequences of the facts that G is 2-connected, $G\not\trianglerighteq_s K_{2,3}$ and $G\not\cong K_4$. Therefore the proofs are omitted. For the proof of (c), assume that $(v, v_1), (v, v_2)\in E(G)\setminus E(C)$ $(v_1\neq v_2)$ and $(v_1, v_2)\notin E(C)$. Then we can find $x_i\in V(G)\setminus\{v, v_1, v_2\}$ $(i=1, 2, 3)$ such that $v_1, x_1, v_2, x_2, v, x_3$

is the cyclic order in which these six vertices appear on C. Then, clearly, we can find an immersion $\varphi: K_{2,3} \to G$ mapping the vertices of $K_{2,3}$ onto $\{v_1, v_2, x_1, x_2, x_3\}$. Hence (c) follows. But then (d) is an immediate consequence of (c). ▮

For each integer $n \geqq 2$, let F_n be the graph with $V(F_n) = \{1, 2, \ldots, n\}$ and $E(F_n) = \{(i, j): 1 \leqq |i-j| \leqq 2\}$ (see Fig. 1). (It is not hard to verify that $F_n \not\geqq K_{2,3}$, $n \geqq 2$; however, this will not be used for the proof of Theorem 1.)

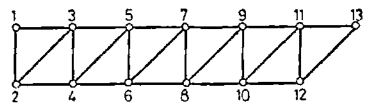

Fig. 1. F_{13}

As a consequence of Lemma 4 we get the following.

Lemma 5. *Let G and C be as in Lemma 4. Let P be a component of $G-E(C)$ such that $n := |V(P)| \geqq 2$ and let S be the graph that is induced in G by the vertices of P. Then*

(a) $S \cong F_n$,

(b) *there are exactly two branches B_1, B_2 of S in G,*

(c) $B_i \cap S \cong K_2$ $(i = 1, 2)$,

(d) $\gamma(v, B_i) = 2$ *for each* $v \in V(B_i \cap S)$ $(i = 1, 2)$.

Proof. By Lemma 4 (d), $\gamma(v, G-E(C)) \leqq 2$ $(v \in V(G))$ and therefore P is a path or a cycle. If $n \leqq 3$, then the lemma clearly follows from Lemma 4. Let $n \geqq 4$. If P were a cycle, say, $P = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, then $(v_1, v_3), (v_2, v_4) \in E(C)$ by Lemma 4 (c); this contradicts Lemma 4 (a), and therefore P is a path $(v_1, v_2, \ldots, v_n)$. Let S' be the graph with $V(S') = \{v_1, v_2, \ldots, v_n\}$, $E(S') = \{(v_i, v_j): 1 \leqq |i-j| \leqq 2\}$ and let $P_1 = (v_1, v_3, \ldots)$ (respectively, $P_2 = (v_2, v_4, \ldots)$) be the path $\subseteqq S'$ which is formed by the vertices with odd (even) subscripts. Then, by Lemma 4 (c), $P_i \subseteqq C$ $(i = 1, 2)$. Further (by Lemma 4 (a)) the paths P_1, P_2 are placed on C such that v_1, v_3, v_4, v_2 is the cyclic order in which these vertices appear on C. From this one easily concludes that Lemma 5 holds. (We leave the details to the reader.) ▮

Let G and S be as in Lemma 5. Then S is called a *separator of* G.

Lemma 6. *Let $\mathscr{F}$ be the class of graphs F such that $F \cong F_n$ for some $n \geqq 2$. Then $\mathscr{F}$ is well-behaved.*

Proof. Let $(Q, \leqq)$ be a well-quasi-order and let (A_i, λ_i) $(i = 1, 2, \ldots)$ be a sequence of labelled graphs such that $A_i \cong F_{n(i)}$ and $\lambda_i: V(A_i) \to Q$. Denote the vertices of A_i by $v_{i,1}, \ldots, v_{i,n(i)}$, where $v_{i,k}$ is the vertex that, under an isomorphism $A_i \to F_{n(i)}$, corresponds to the vertex $k \in V(F_{n(i)})$, $i = 1, 2, \ldots$. It is easy to verify (by Lemma 1(a), (c)) that we can find $i, j \in \mathbb{N}$ $(i < j)$ such that there is a mapping $\varphi: V(A_i) \to V(A_j)$ with the following properties. (i) If k is odd and $\varphi(v_{i,k}) = v_{j,m}$, then m is odd, (ii) if k is even and $\varphi(v_{i,k-1}) = v_{j,m}$, then $\varphi(v_{i,k}) = v_{j,m+1}$, (iii) if $\varphi(v_{i,k}) = v_{j,m}$, $\varphi(v_{i,\varkappa}) = v_{j,\mu}$ for $k < \varkappa$, then $m < \mu$, (iv) $\lambda_i(v) \leqq \lambda_j(\varphi(v))$ for each $v \in V(A_i)$. It is left to the reader to construct the corresponding immersion $\psi: A_i \to A_j$ such that $\psi(v) = \varphi(v)$ for each $v \in V(A_i)$. ▮

Lemma 7. *The class $\mathscr{K}$ of 2-connected graphs G with $K_4 \not\leqq G \not\geqq K_{2,3}$ is well-behaved.*

Proof. Let $(Q, \leqq)$ be a well-quasi-order and let $\mathscr{K}'$ be the class of "double rooted" labelled graphs (A, a^1, a^2, λ), where $A \in \mathscr{K}$, $\lambda: V(A) \to Q$, and $a^1, a^2 \in V(A)$ is an ordered pair of roots with $(a^1, a^2) \in E(A)$. (Thus we shall distinguish between (A, a^1, a^2, λ) and (A, a^2, a^1, λ).) For $(A, a^1, a^2, \lambda), (B, b^1, b^2, \mu) \in \mathscr{K}'$, we write $(A, a^1, a^2, \lambda) \lesssim (B, b^1, b^2, \mu)$ if there is an immersion $\varphi: A \to B$ such that $\varphi(a^k) = b^k$ $(k=1,2)$ and $\lambda(v) \leqq \mu(\varphi(v))$ for each $v \in V(A)$. Clearly, this is a quasi-order on $\mathscr{K}'$, and our lemma is proved if we show that $\mathscr{K}'$ is wqo by $\lesssim$. Thus assume the contrary and let $(A_i, a_i^1, a_i^2, \lambda_i)$ $(i=1,2,\ldots)$ be a minimal bad sequence of $(\mathscr{K}', \lesssim)$. Let $N_1 \subseteq \mathbf{N}$ be the set of all i such that the edge (a_i^1, a_i^2) is in a separator of A_i.

Case 1. N_1 is infinite.

For each $i \in N_1$, let S_i be the separator of A_i that contains (a_i^1, a_i^2). By Lemma 5 (b), (c) there are just two branches $B_{i,1}, B_{i,2}$ of S_i an A_i and $B_{i,k} \cap S_i \cong K_2$ $(k=1,2)$. Denote the vertices of $B_{i,k} \cap S_i$ by $b_{i,k}^1$ and $b_{i,k}^2$ such that there are two disjoint $b_{i,k}^t, a_i^t$-paths $Q_{i,k}^t \subseteq S_i$ $(1 \leqq t,\ k \leqq 2;\ i \in N_1)$. (This is possible since, by Lemma 5(a), S_i is either 2-connected or isomorphic to K_2.) For all $i \in N_1$, $k \in \{1,2\}$, let $\mu_{i,k}: V(B_{i,k}) \to Q$ be defined by $\mu_{i,k}(b_{i,k}^t) := \lambda_i(a_i^t)$ $(t=1,2)$, $\mu_{i,k}(v) := \lambda_i(v)$ for $v \neq b_{i,k}^t$ $(t=1,2)$. By Lemma 5(d), $\gamma(b_{i,k}^t, B_{i,k} - (b_{i,k}^1, b_{i,k}^2)) = 1$ $(1 \leqq t,\ k \leqq 2;\ i \in N_1)$. From this one easily finds (using $Q_{i,k}^1, Q_{i,k}^2$) that $(B_{i,k}, b_{i,k}^1, b_{i,k}^2, \mu_{i,k}) \lesssim (A_i, a_i^1, a_i^2, \lambda_i)$ $(k=1,2;\ i \in N_1)$. Clearly, $(B_{i,k}, b_{i,k}^1, b_{i,k}^2, \mu_{i,k}) \in \mathscr{K}'$ and $|E(B_{i,k})| < |E(A_i)|$ $(k=1,2;\ i \in N_1)$. Thus we conclude from Lemma 2 (b) that

(1) *for each $k=1,2$, the set $\{(B_{i,k}, b_{i,k}^1, b_{i,k}^2, \mu_{i,k}): i \in N_1\}$ is wqo by $\lesssim$.*

Let $M := \{1, 2, \ldots, 6\}$ and let M' be the set of subsets of M. Let $\leqq_{tr}$ be the trivial well-quasi-order on M'. For all $i \in N_1$, $v \in V(S_i)$, define an (additional) label $\sigma_i(v) \in M'$ as follows. For each $t \in \{1,2\}$, let $t \in \sigma_i(v)$ iff $v = b_{i,1}^t$, let further $2+t \in \sigma_i(v)$ iff $v = b_{i,2}^t$ and let $4+t \in \sigma_i(v)$ iff $v = a_i^t$. Let $\mathscr{F}$ be as in Lemma 6. Then, by Lemma 1 (a) and Lemma 6

(2) $$\mathscr{F}(Q \times M') \text{ is wqo by } \lesssim_{(Q \times M', \leqq \wedge \leqq_{tr})}.$$

From (1), (2) together with the Lemmas 1(a), 5(a) and the definition of $\sigma_i(v)$, one concludes that there are $j, m \in N_1$ $(j<m)$ such that

(3) $$(B_{j,k}, b_{j,k}^1, b_{j,k}^2, \mu_{j,k}) \lesssim (B_{m,k}, b_{m,k}^1, b_{m,k}^2, \mu_{m,k}), \quad k = 1, 2,$$

(4) *there is an immersion $\psi: S_j \to S_m$ such that $\psi(a_j^t) = a_m^t$, $\psi(b_{j,k}^t) = b_{m,k}^t$ $(1 \leqq k,\ t \leqq 2)$, and $\lambda_j(v) \leqq \lambda_m(\psi(v))$ for each $v \in V(S_j)$.*

Let $\varphi_k: B_{j,k} \to B_{m,k}$ be an immersion according to (3), $k=1,2$. Since $\varphi_k(b_{j,k}^t) = b_{m,k}^t = \psi(b_{j,k}^t)$ $(1 \leqq t,\ k \leqq 2)$, we can define an immersion $\varphi: A_j \to A_m$ which coincides with ψ on S_j and with φ_k on $B_{j,k} - (b_{j,k}^1, b_{j,k}^2)$, $k=1,2$. Consequently, $(A_j, a_j^1, a_j^2, \lambda_j) \lesssim (A_m, a_m^1, a_m^2, \lambda_m)$ $(j<m)$, which contradicts the choice of the sequence $(A_i, a_i^1, a_i^2, \lambda_i)$, $i=1,2,\ldots$. This settles Case 1.

Let $N_2 \subseteq \mathbf{N}$ be the set of all i for which there exists an edge $(b_i^1, b_i^2) \in E(A_i - (a_i^1, a_i^2))$ such that $A_i - \{(a_i^1, a_i^2), (b_i^1, b_i^2)\}$ consists of two non-trivial components.

Case 2. N_2 is infinite.

Let (b_i^1, b_i^2) be as in the definition of N_2 and denote the components of $A_i - \{(a_i^1, a_i^2), (b_i^1, b_i^2)\}$ by B_i^1, B_i^2; choose the notation such that $a_i^k, b_i^k \in V(B_i^k)$ $(k=1, 2;\ i\in N_2)$. (This is possible since A_i is 2-connected.) For all $i\in N_2$, $k\in\{1, 2\}$, let A_i^k be the graph with $V(A_i^k)=V(B_i^k)\cup\{a_i^l\}$ and $E(A_i^k)=E(B_i^k)\cup\{(a_i^k, a_i^l), (b_i^k, a_i^l)\}$, where $l\in\{1, 2\}$, $l\neq k$. (Note that $a_i^k\neq b_i^k$ since B_i^k is non-trivial and A_i is 2-connected.) Each A_i^k is 2-connected and clearly $K_4 \not\leq A_i^k \not\geq K_{2,3}$; moreover $|E(A_i^k)|<|E(A_i)|$. Let λ_i^k be the restriction of λ_i to $V(A_i^k)$. Clearly, $(A_i^k, a_i^1, a_i^2, \lambda_i^k) \leqq (A_i, a_i^1, a_i^2, \lambda_i)$ $(k=1, 2,\ i\in N_2)$. Consequently, by Lemma 1(a) and Lemma 2(b), there are $j, m\in N_2$ $(j<m)$ such that

$$(5)\qquad (A_j^k, a_j^1, a_j^2, \lambda_j^k) \leqq (A_m^k, a_m^1, a_m^2, \lambda_m^k), \quad k = 1, 2.$$

Let $\varphi_k\colon A_j^k \to A_m^k$ be an immersion according to (5), $k=1, 2$. Define the immersion $\varphi\colon A_j \to A_m$ as follows. Let φ coincide with φ_k on B_j^k, $k=1, 2$, and let further $\varphi((a_j^1, a_j^2))=(a_m^1, a_m^2)$. It remains to define $\varphi((b_j^1, b_j^2))$. For this, note that the path $\varphi_k((b_j^k, a_j^l))$ $(l\neq k)$ must contain the edge (b_m^k, a_m^l), and therefore $P_k:=\varphi_k((b_j^k, a_j^l))\cap \cap B_m^k$ is a $\varphi_k(b_j^k), b_m^k$-path, $k=1, 2$. Thus we can define $\varphi((b_j^1, b_j^2)):=P_1\cup(b_m^1, b_m^2)\cup \cup P_2$. Hence $(A_j, a_j^1, a_j^2, \lambda_j)\leqq(A_m, a_m^1, a_m^2, \lambda_m)$ $(j<m)$. This contradiction settles Case 2.

Case 3. $N_1\cup N_2$ is finite.

Let C_i be the Hamilton cycle of A_i, $i\in \mathbf{N}$ (see Lemma 4 (b)). Recall that $i\notin N_1$ implies that $(a_i^1, a_i^2)\in E(C_i)$, and let $(x_i^1=a_i^1, x_i^2, \dots, x_i^{t(i)}=a_i^2)$ be the path $C_i-(a_i^1, a_i^2)$ $(i\in \mathbf{N}\setminus N_1)$. Let $N_3\subseteq \mathbf{N}$ be the set of i for which A_i is a triangle. Clearly, N_3 must be finite. Let $N_4:=\mathbf{N}\setminus(N_1\cup N_2\cup N_3)$.

(6) *For each $i\in N_4$, at least one of the following three conditions holds.*

(i) $(x_i^1, x_i^{t(i)-1})\in E(A_i)$, (ii) $(x_i^2, x_i^{t(i)})\in E(A_i)$, (iii) $(x_i^2, x_i^{t(i)-1})\in E(A_i)$.

Proof. Assume that, for a certain $i\in N_4$, none of these conditions holds. If $\gamma(x_i^1, A_i)=\gamma(x_i^2, A_i)=2$, then (since $i\notin N_3$) $A_i-\{(a_i^1, a_i^2), (x_i^2, x_i^3)\}$ would consist of two non-trivial components, in contradiction to $i\notin N_2$. Hence we can pick an edge $(x_i^\nu, x_i^\mu)\in E(A_i)\setminus E(C_i)$ with $\nu\in\{1, 2\}$ such that μ is as large as possible. Note that $\mu\leqq t(i)-2$ since, otherwise, (i), (ii) or (iii) would hold. Further, one concludes from Lemma 4 (c) and $i\notin N_1$ that $(x_i^\mu, x_i^j)\notin E(A_i)$ for each $j=\mu+2, \dots, t(i)$. From this together with the choice of μ, one concludes that $A_i-\{(a_i^1, a_i^2), (x_i^\mu, x_i^{\mu+1})\}$ consists of two non-trivial components, which contradicts $i\notin N_2$. ∎

For each $i\in N_4$, let $b_{i,1}^1:=x_i^1$, $b_{i,1}^2:=x_i^{t(i)-1}$ if (i) holds, let $b_{i,1}^1:=x_i^2$, $b_{i,1}^2:=x_i^{t(i)}$ if (ii) holds, and let $b_{i,1}^1:=x_i^2$, $b_{i,1}^2:=x_i^{t(i)-1}$ if neither (i) nor (ii) holds. Then $(b_{i,1}^1, b_{i,1}^2)\in E(A_i)\setminus E(C_i)$. (For this, recall that A_i is not a cycle of length 4 since, otherwise, $i\in N_2$.) Let S_i be the separator of A_i that contains $(b_{i,1}^1, b_{i,1}^2)$; then there are exactly two branches $B_{i,1}, B_{i,2}$ of S_i in A_i, and one branch, say $B_{i,1}$, contains (a_i^1, a_i^2). (Thus $B_{i,1}$ is a cycle of length three or four.) Denote the vertices of $B_{i,2}\cap S_i$ by $b_{i,2}^1, b_{i,2}^2$ such that there are two disjoint $b_{i,2}^t, a_i^t$-paths in $S_i\cup B_{i,1}$ $(t=1, 2;\ i\in N_4)$. As in Case 1, one finds that (with $\mu_{i,2}$ defined as in Case 1)

(7) *the set $\{(B_{i,2}, b_{i,2}^1, b_{i,2}^2, \mu_{i,2})\colon i\in N_4\}$ is wqo by $\leqq$.*

Clearly, there is an infinite subset N_5 of N_4 such that, for each $i, j \in N_5$, there is an isomorphism $\alpha: B_{i,1} \to B_{j,1}$ such that $\alpha(b^t_{i,1}) = b^t_{j,1}$ and $\alpha(a^t_i) = a^t_j$, $t = 1, 2$. Similar to the discussion of Case 1, one finds that there are $j, m \in N_5$ $(j < m)$ such that (i) $(B_{j,2}, b^1_{j,2}, b^2_{j,2}, \mu_{j,2}) \leqq (B_{m,2}, b^1_{m,2}, b^2_{m,2}, \mu_{m,2})$, (ii) there is an immersion $\psi: S_j \to S_m$ such that $\psi(b^t_{j,k}) = b^t_{m,k}$ $(1 \leqq k, t \leqq 2)$ and $\lambda_j(v) \leqq \lambda_m(\psi(v))$ for each $v \in V(S_j)$, (iii) $\lambda_j(a^t_j) \leqq \lambda_m(a^t_m)$ $(t = 1, 2)$. From this, it follows that $(A_j, a^1_j, a^2_j, \lambda_j) \leqq (A_m, a^1_m, a^2_m, \lambda_m)$ $(j < m)$, which is a contradiction. Hence Case 3 is settled. ▮

Proof of Theorem 1. Let $\mathscr{K}$ be the class of *connected* graphs G with $G \not\geqq K_{2,3}$. By Lemma 1(b) it is sufficient to show that $\mathscr{K}$ is wqo by $\leqq$. Let $\mathscr{K}_1$ be the class of rooted graphs (A, a) with $A \in \mathscr{K}$, $a \in V(A)$. For each $n \in \mathbb{N}$ let us define a relation $\leqq_n$ on $\mathscr{K}_1$ as follows. For $(A, a), (B, b) \in \mathscr{K}_1$, let $(A, a) \leqq_n (B, b)$ if there exists an immersion $\varphi: A \to B$ together with n edge-disjoint $\varphi(a), b$-paths $P_i \subseteq B$ $(1 \leqq i \leqq n)$ such that $E(P_i) \cap E(\varphi(A)) = \emptyset$ and $V(P_i) \cap \{\varphi(v): v \in V(A),\ v \neq a\} = \emptyset$ $(1 \leqq i \leqq n)$. Further, let us write $(A, a) \leqq (B, b)$ if there is an immersion $\varphi: A \to B$ with $\varphi(a) = b$. Note that in the definition of $\leqq_n$ the paths P_i are allowed to have length 0; hence $(A, a) \leqq (B, b)$ implies $(A, a) \leqq_n (B, b)$ for each $n \in \mathbb{N}$. One easily finds that the relations $\leqq_n$ and $\leqq$ are quasi-orders. (In the present proof, we shall only be concerned with $\leqq_1$, $\leqq_2$ and $\leqq$.) Theorem 1 is proved if we show that $\mathscr{K}_1$ is wqo by $\leqq_1$. Thus assume the contrary and let (A_i, a_i) $(i = 1, 2, \ldots)$ be a minimal bad sequence of $(\mathscr{K}_1, \leqq_1)$. Let $\mathscr{K}_2$ be the class of rooted graphs (B, b) such that B is connected and $(B, b) \leqq_1 (A_i, a_i)$ for some $i \in \mathbb{N}$. Then $\mathscr{K}_2$ is not wqo by $\leqq_2$ (since (A_i, a_i) $(i = 1, 2, \ldots)$ is a bad sequence of $(\mathscr{K}_2, \leqq_2)$. Let (B_i, b_i) $(i = 1, 2, \ldots)$ be a minimal bad sequence of $(\mathscr{K}_2, \leqq_2)$. Let $\mathscr{K}_3$ be the class of rooted graphs (C, c) such that C is connected and $(C, c) \leqq_2 (B_i, b_i)$ for some $i \in \mathbb{N}$. Then $\mathscr{K}_3$ is not wqo by $\leqq$. Let (C_i, c_i) $(i = 1, 2, \ldots)$ be a minimal bad sequence of $(\mathscr{K}_3, \leqq)$. Let f, g be functions $\mathbb{N} \to \mathbb{N}$ such that $(C_i, c_i) \leqq_2 (B_{f(i)}, b_{f(i)})$ and $(B_i, b_i) \leqq_1 (A_{g(i)}, a_{g(i)})$ $(i = 1, 2, \ldots)$, and define $h(i) := g(f(i))$ $(i = 1, 2, \ldots)$. Moreover, we may assume that the graphs C_i $(i = 1, 2, \ldots)$ are disjoint.

Let $N_1 := \{i \in \mathbb{N}: c_i$ is a cut vertex of $C_i\}$ and $N_2 := \{i \in \mathbb{N}: \gamma(c_i, C_i) = 1\}$. We claim that

(1) $N_1 \cup N_2$ *is finite.*

Proof. First, assume that N_1 is infinite. For each $i \in N_1$, let $C^k_i \subseteq C_i$ $(k = 1, 2)$ such that $C^1_i \cap C^2_i = c_i$, $C^1_i \cup C^2_i = C_i$ and $|E(C^k_i)| < |E(C_i)|$ $(k = 1, 2)$. Then $(C^k_i, c_i) \leqq (C_i, c_i)$. Moreover, C^k_i is connected, $(C^k_i, c_i) \leqq_2 (B_{f(i)}, b_{f(i)})$, and therefore $C^k_i \in \mathscr{K}_3$. Hence by the minimality of the sequence (C_i, c_i) $(i = 1, 2, \ldots)$ together with Lemmas 1 (a) and 2 (b) there are $j, m \in \mathbb{N}$ $(j < m)$ such that $(C^k_j, c_j) \leqq (C^k_m, c_m)$ $(k = 1, 2)$. Hence $(C_j, c_j) \leqq (C_m, c_m)$ $(j < m)$, in contradiction to the choice of the sequence (C_i, c_i) $(i = 1, 2, \ldots)$. Thus N_1 is finite. Next assume that N_2 is infinite. For each $i \in N_2$, let $C'_i := C_i - c_i$ and let $c'_i \in V(C'_i)$ be the unique neighbour of c_i. Then $(C'_i, c'_i) \leqq_1 (C_i, c_i) \leqq_1 (A_{h(i)}, a_{h(i)})$, $(C'_i, c'_i) \in \mathscr{K}_1$ and $|E(C'_i)| < |E(C_i)| \leqq |E(A_{h(i)})|$ $(i \in N_2)$. Hence, by the minimality of the sequence (A_i, a_i) $(i = 1, 2, \ldots)$ together with Lemma 2 (b), there are $j, m \in N_2$ $(j < m)$ such that $(C'_j, c'_j) \leqq_1 (C'_m, c'_m)$. From this one easily concludes that $(C_j, c_j) \leqq (C_m, c_m)$, which is a contradiction. Hence N_2 is finite and thus (1) is proved. ▮

Let $N_3 := \mathbf{N} \setminus (N_1 \cup N_2)$. Let D_i be the unique block of C_i with $c_i \in D_i$ $(i \in N_3)$. For each $i \in N_3$ and each $z \in V(D_i)$, let $\mathscr{Y}_z$ be the set of rooted graphs (Y, z) such that Y is a branch of D_i in C_i with $z \in Y$. Further, if $\mathscr{Y}_z \neq \emptyset$, let U_z be the union of the graphs Y with $(Y, z) \in \mathscr{Y}_z$; if $\mathscr{Y}_z = \emptyset$, let $U_z = z$. If $\gamma(z, Y) \leqq 2$ for each $(Y, z) \in \mathscr{Y}_z$, then z is called a *vertex of type* 1; otherwise z is called a *vertex of type* 2 $(z \in V(D_i)$, $i \in N_3)$. Let $\mathscr{Z}_t$ be the set of rooted graphs (U_z, z) such that z is a vertex of type t $(t = 1, 2)$. We claim that

(2) $\mathscr{Z}_1$ *is wqo by* $\trianglelefteq$.

(3) $\mathscr{Z}_2$ *is wqo by* $\trianglelefteq_2$.

Proof of (2). Let $\mathscr{Y} = \bigcup \mathscr{Y}_z$ where the union is taken over all vertices z that are of type 1. First we show that $\mathscr{Y}$ is wqo by $\trianglelefteq$. This is clear if $\mathscr{Y} = \emptyset$. Hence assume that $\mathscr{Y} \neq \emptyset$ and let $(Y, z) \in \mathscr{Y}$, $z \in V(D_i)$. Then one easily finds that $(Y, z) \trianglelefteq (C_i, c_i)$. (For this use the facts that $\gamma(z, Y) \leqq 2$ and that D_i is 2-connected.) Clearly, $|E(Y)| < |E(C_i)|$ and $(Y, z) \in \mathscr{K}_3$. Thus, by the minimality of the sequence (C_i, c_i) $(i = 1, 2, \ldots)$ together with Lemma 2(b), $\mathscr{Y}$ is wqo by $\trianglelefteq$. Hence, by Lemma 1 (b), $\mathscr{Y}^*$ is wqo by $\trianglelefteq^*$. One easily verifies that this implies (2). ▮

Proof of (3). For $(U_z, z) \in \mathscr{Z}_2$, $z \in V(D_i)$, one easily finds that $(U_z, z) \trianglelefteq_2 (C_i, c_i) \trianglelefteq_2 (B_{f(i)}, b_{f(i)})$, $(U_z, z) \in \mathscr{K}_2$ and $|E(U_z)| < |E(C_i)| \leqq |E(B_{f(i)})|$. Hence (3) follows from the minimality of the sequence (B_i, b_i) $(i = 1, 2, \ldots)$ together with Lemma 2 (b). For each $z \in V(D_i)$, define the label $\lambda_i(z) \in \mathscr{Z} := \mathscr{Z}_1 \cup \mathscr{Z}_2$ by $\lambda_i(z) := (U_z, z)$ $(i \in N_3)$. ▮

For $(Z, z), (Z', z') \in \mathscr{Z}$, let $(Z, z) \trianglelefteq^+ (Z', z')$ if and only if either $(Z, z), (Z', z') \in \mathscr{Z}_1$ and $(Z, z) \trianglelefteq (Z', z')$, or $(Z, z), (Z', z') \in \mathscr{Z}_2$ and $(Z, z) \trianglelefteq_2 (Z', z')$. Clearly, by (2), (3) and since $\mathscr{Z}_1 \cap \mathscr{Z}_2 = \emptyset$, $\mathscr{Z}$ is wqo by $\trianglelefteq^+$. Further, let $S = \{0, 1\}$ and let $\leqq_{\mathrm{tr}}$ be the trivial wqo on S. For $z \in V(D_i)$, let $\mu_i(z) = 1$ if $z = c_i$, and $\mu_i(z) = 0$ otherwise $(i \in N_3)$. Let $\mathscr{D}$ be the class of 2-connected graphs G with $G \not\trianglerighteq K_{2,3}$. Then one finds as a consequence of Lemma 7 that $\mathscr{D}$ is well-behaved. Consequently (since each D_i is in $\mathscr{D}$), we can find $j, m \in N_3$ $(j < m)$ such that there is an immersion $\varphi: D_j \to D_m$ with $\lambda_j(z) \trianglelefteq^+ \lambda_m(\varphi(z))$ and $\mu_j(z) \leqq_{\mathrm{tr}} \mu_m(\varphi(z))$. Hence, if $z \in D_j$ is a vertex of type 1, then there exists an immersion ψ_z: $U_z \to U_{\varphi(z)}$ with $\psi_z(z) = \varphi(z)$, and, if $z \in D_j$ is a vertex of type 2, then there exists an immersion ψ_z: $U_z \to U_{\varphi(z)}$ together with two $\psi_z(z)$, $\varphi(z)$-paths $P_{z,1}, P_{z,2} \subseteq U_{\varphi(z)}$ according to the definition of $\trianglelefteq_2$. Further, $\varphi(c_j) = c_m$.

If $z \in V(D_j)$ is of type 2, then it follows from Lemma 3 that there are just two edges of D_j that are incident with z. Call these edges $e_{z,1}, e_{z,2}$. Now we can define an immersion $\Phi: C_j \to C_m$ as follows. For each U_z $(z \in V(D_j))$, let Φ coincide with ψ_z. Let $e = (x, y) \in E(D_j)$. If both x and y are of type 1, then let $\Phi(e) := \varphi(e)$. If x is of type 1 and y is of type 2, then $e = e_{y,t}$ for a certain $t \in \{1, 2\}$; in this case let $\Phi(e) := \varphi(e) \cup P_{y,t}$. If both x and y are of type 2, then $e = e_{x,s} = e_{y,t}$ for certain $s, t \in \{1, 2\}$; in this case let $\Phi(e) := \varphi(e) \cup P_{x,s} \cup P_{y,t}$. The reader may convince himself that this defines an immersion $\Phi: C_j \to C_m$. Moreover, c_j and c_m are of type 1 (because $j, m \notin N_1$), and therefore $\Phi(c_j) = \varphi(c_j) = c_m$. Hence $(C_j, c_j) \trianglelefteq (C_m, c_m)$ $(j < m)$, in contradiction to the choice of the sequence (C_i, c_i) $(i = 1, 2, \ldots)$. ▮

4. Another result on well-quasi-ordering by immersion

For a class $\mathscr{B}$ of graphs, let us denote by $[\mathscr{B}]$ the class of connected graphs G such that each block of G is a member of $\mathscr{B}$.

Theorem 2. *For* $k=1, 2, \dots,$ *let* $\mathscr{G}_k$ *be the class of k-regular, k-edge-connected graphs, and let* $\mathscr{G}:=\bigcup_{k=1}^{\infty}\mathscr{G}_k$. *Let* $\mathscr{B}$ *be a subclass of* $\mathscr{G}$ *that is well-behaved. Then* $[\mathscr{B}]$ *is wqo by* $\leqq$.

For example, the following subclasses of $\mathscr{G}$ are well-behaved (by Lemma 1). (i) The class of complete graphs K_n $(n\geqq 2)$, (ii) the class of cycles, (iii) the class of complete bipartite graphs $K_{n,n}$ $(n\geqq 1)$. Clearly, the union of a finite number of well-behaved classes is also well-behaved; thus the result mentioned in the abstract follows.

Proof of Theorem 2. For a class $\mathscr{A}$ of graphs and a quasi-order $(Q, \leqq)$ let us denote by $\mathscr{A}_Q$ the class of *rooted labelled graphs* (A, a, λ) where $A\in\mathscr{A}$, $a\in V(A)$ and $\lambda\colon V(A)\to Q$. For $(A, a, \lambda), (B, b, \mu)\in\mathscr{A}_Q$, let $(A, a, \lambda)\leqq_{(Q,\leqq)}(B, b, \mu)$ if there exists an immersion $\varphi\colon A\to B$ such that $\lambda(v)\leqq\mu(\varphi(v))$ for each $v\in V(A)$ and $\varphi(a)=b$. ($\leqq_{(Q,\leqq)}$ is a quasi-order on $\mathscr{A}_Q$.) First, we shall prove the theorem under the additional assumption that $\mathscr{B}\cap\mathscr{G}_k\neq\emptyset$ holds only for a finite number of k. In order to make induction work, we shall, in fact, prove the following.

$(*)$ *If* $\mathscr{B}\subseteqq\bigcup_{j=1}^{t}\mathscr{G}_{k_j}$ *for certain integers* $k_1<k_2<\dots<k_t$ $(t\in\mathrm{N})$ *and if* $(Q, \leqq)$ *is a well-quasi-order, then* $[\mathscr{B}]_Q$ *is wqo by* $\leqq_{(Q,\leqq)}$.

Assume the contrary. Let s be the least positive integer t for which $(*)$ is false, and pick $k_1<\dots<k_s$, $\mathscr{B}\subseteqq\bigcup_{j=1}^{s}\mathscr{G}_{k_j}$ and a well-quasi-order $(Q, \leqq)$ such that $[\mathscr{B}]_Q$ is not wqo by $\leqq_{(Q,\leqq)}$. For simplicity, we write $\leqq_0$ for $\leqq_{(Q,\leqq)}$. Let (A_i, a_i, λ_i) $(i=1, 2, \dots)$ be a minimal bad sequence of $([\mathscr{B}]_Q, \leqq_0)$. W.l.o.g. assume that the graphs A_i are disjoint. For each $i\in\mathrm{N}$, let B_i be a fixed block of A_i with $a_i\in B_i$ and let $\bar{A}_i\supseteqq B_i$ be the uniquely determined subgraph of A_i such that (i) $\bar{A}_i$ is the union of B_i with some other blocks of A_i each of which is not in $\mathscr{G}_{k_1}$, (ii) $\bar{A}_i$ is connected, (iii) $\bar{A}_i$ is maximal with these properties.

For $i\in\mathrm{N}$, let Z be a branch of $\bar{A}_i$ in A_i. Then $Z\cap\bar{A}_i$ is a single vertex z. Let $\zeta\colon V(Z)\to Q$ be defined by $\zeta(z):=\lambda_i(a_i)$ and $\zeta(v):=\lambda_i(v)$ for $v\neq z$. Then (Z, z, ζ) is called a *rooted labelled branch of* $\bar{A}_i$ *in* A_i. Let $\mathscr{Z}_i$ be the set of all rooted labelled branches of $\bar{A}_i$ in A_i $(i=1, 2, \dots)$ and $\mathscr{Z}:=\bigcup_{i=1}^{\infty}\mathscr{Z}_i$. Clearly, $\mathscr{Z}\subseteqq[\mathscr{B}]_Q$ and $|E(Z)|<|E(A_i)|$ for each $(Z, z, \zeta)\in\mathscr{Z}_i$ $(i\in\mathrm{N})$. Moreover, one easily verifies that $(Z, z, \zeta)\leqq_0(A_i, a_i, \lambda_i)$ for each $(Z, z, \zeta)\in\mathscr{Z}_i$ $(i\in\mathrm{N})$. (For this note that the uniquely determined block of Z that contains z must be a member of $\mathscr{G}_{k_1}$ and, consequently, $\gamma(z, Z)=k_1$. Further, note that it follows from $k_1\leqq k_j$ $(j=1, \dots, s)$ that we can find k_1 edge-disjoint z, a_i-paths in $\bar{A}_i$. Use these facts to find an appropriate immersion $\varphi\colon Z\to A_i$.) Hence by Lemma 2(b), $\mathscr{Z}$ is wqo by $\leqq_0$, and consequently by

Lemma 1

(1) $$Q\times\mathscr{Z}^* \text{ is wqo by } \leqq\wedge\leqq_0^*.$$

Let $\mathscr{B}':=\mathscr{B}\setminus\mathscr{G}_{k_1}$. If $s=1$, then $\mathscr{B}'=\emptyset$ and the statement (2) below is trivial; otherwise (2) is a consequence of (1) and the minimal choice of s.

(2) $$[\mathscr{B}']_{Q\times\mathscr{Z}^*} \text{ is wqo by } \leqq_{(Q\times\mathscr{Z}^*,\, \leqq\wedge\leqq_0^*)}.$$

Let $\mathscr{U}$ be the union of $[\mathscr{B}']_{Q\times\mathscr{Z}^*}$ with the class of rooted labelled graphs consisting of a single vertex which is labelled by an element of $Q\times\mathscr{Z}^*$. Write $\leqq_1$ for the quasi-order $\leqq_{(Q\times\mathscr{Z}^*,\, \leqq\wedge\leqq_0^*)}$ defined on $\mathscr{U}$. It follows from (1) and (2) that

(3) $$\mathscr{U} \text{ is wqo by } \leqq_1.$$

For $z\in V(\bar{A}_i)$, let $\mu_i(z)\in\mathscr{Z}^*$ be defined as the set of rooted labelled branches of $\bar{A}_i$ in A_i which have z as their root ($i\in\mathrm{N}$). For $x\in V(B_i)$, let U_x be the union of all branches of B_i in $\bar{A}_i$ which contain x, provided that at least one such branch exists; otherwise, let $U_x=x$. Further, let $\xi_x\colon V(U_x)\to Q\times\mathscr{Z}^*$ be defined by $\xi_x(z):=$ $:=(\lambda_i(z), \mu_i(z))$ $(z\in V(U_x))$. Then $(U_x, x, \xi_x)\in\mathscr{U}$ $(x\in V(B_i),\ i\in\mathrm{N})$. Consequently, by (3) and since $\mathscr{B}$ is well-behaved, we can find $j, m\in\mathrm{N}$ $(j<m)$ together with an immersion $\psi\colon B_j\to B_m$ such that (i) $(U_x, x, \xi_x)\leqq_1 (U_{\psi(x)}, \psi(x), \xi_{\psi(x)})$ for each $x\in V(B_j)$, and (ii) $\psi(a_j)=a_m$. (For (ii), use the usual labelling technique which assigns 1 to the root a_i and 0 to all other vertices of B_i, $i=1, 2, \dots$; see also the proof of Theorem 1.) Now, it is a matter of routine to verify that this implies $(A_j, a_j, \lambda_j)\leqq_0(A_m, a_m, \lambda_m)$. This contradiction proves $(*)$.

In order to complete the proof of Theorem 2, let $G_i\in[\mathscr{B}]$ $(i=1, 2, \dots)$. Let $\mathscr{C}$ be the class of graphs that occur as a block of one of the graphs G_i $(i\geqq 2)$. If $\mathscr{C}\cap\mathscr{G}_k\neq\emptyset$ holds only for a finite number of k, then one concludes from $(*)$ that G_i $(i=1, 2, \dots)$ is good. If $\mathscr{C}\cap\mathscr{G}_k\neq\emptyset$ for infinitely many k, then $\mathscr{C}$ contains members of arbitrarily high regularity, and thus we conclude from a theorem of Mader [8] that there is a $C\in\mathscr{C}$ which contains a subdivision of a complete graph with $|V(G_1)|$ vertices. Hence $G_1\leqq G_i$ for some $i\geqq 2$, which proves that G_i $(i=1, 2, \dots)$ is good. ∎

Added in proof: Recently Robertson and Seymour proved that even the class of all finite graphs is wqo by the minor inclusion relation. In particular this implies that Vázsonyi's second conjecture is also true.

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